

COMPUTER MECHANIZATION OF SIX - DEGREE OF  
FREEDOM FLIGHT EQUATIONS

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# COMPUTER MECHANIZATION OF SIX-DEGREE OF FREEDOM FLIGHT EQUATIONS

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## 1. Introduction

Solution of the six-degree-of-freedom flight equations for aircraft and missiles continues to represent one of the most important application areas for analog, hybrid, and digital computer systems. Important computer requirements such as accuracy and speed are dependent very much on the choice of axis system for the translation equations of motion. In this connection it is well known that the flight-path axis system makes much lower accuracy and speed demands on the computer than does the body-axis system [ 1, 2 ]. Despite this a number of current computer mechanizations continue to use body axes for solving the translational equations of motion. Because of this, unnecessary demands of accuracy or frequency response are placed on the computer and many mechanizations which could be all analog or all digital have shifted to hybrid implementation. Even if the mechanization is hybrid from the outset, there is considerable advantage to be gained by using an efficient axis system. The purpose of this paper is to point out again the advantages of flight path axes and to summarize the overall equation requirements for solving the six-degree of freedom flight equations.

## 2. Body-Axis Translational Equations

For comparison purposes we present first the body-axis translational equations. The body axes  $x_b$ ,  $y_b$ , and  $z_b$  are defined as a right-hand set fixed to the vehicle with the  $x_b$  axis along the longitudinal axis and the  $z_b$  axis directed downward for normal level flight. The components of the total vehicle velocity vector  $\bar{V}_p$  along the  $x_b$ ,  $y_b$  and  $z_b$  axes, respectively, are  $U_b$ ,  $V_b$ , and  $W_b$  (see Fig. 2.1). The components of the body-axis angular velocity vector  $\bar{\Omega}$  (and hence the vehicle angular velocity vector) along the  $x_b$ ,  $y_b$ , and  $z_b$  axes are  $P_b$ ,  $Q_b$  and  $R_b$ , i. e., roll rate, pitch rate, and yaw rate, respectively. Here we assume  $\bar{V}_p$  and  $\bar{\Omega}$  represent vehicle translational and rotational velocity vectors as viewed from an inertial (non-accelerating) frame of reference. If we denote the external forces along a set of coordinate axes by  $X$ ,  $Y$ , and  $Z$ , respectively, then Euler's translational equations of motion, obtained by summing forces along the coordinate axes, are the following:

$$m(\dot{U} - VR + WQ) = X \quad (2.1)$$

$$m(\dot{V} - WP + UR) = Y \quad (2.2)$$

$$m(\dot{W} - UQ + VP) = Z \quad (2.3)$$

where  $m$  is the mass of the vehicle. The inefficiency of these equations in body axes is immediately apparent when one considers the approximate size of the various terms. Let the vehicle be, say, a Mach 2 aircraft with  $V_{\max} = 2000$  ft/sec. A reasonable upper limit on pitch-rate  $Q_b$

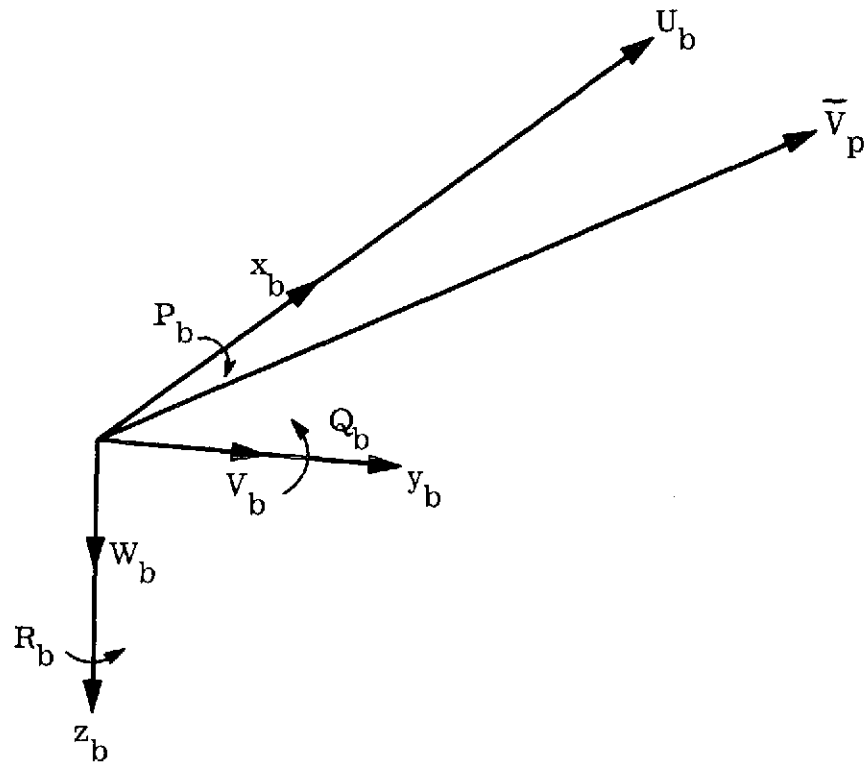


Figure 2.1 Body Axes  $x_b$ ,  $y_b$ ,  $z_b$  with Velocity Components  $U_b$ ,  $V_b$ , and  $W_b$  and Angular Velocity Components  $P_b$ ,  $Q_b$ , and  $R_b$  Respectively.

might be 2 radians/sec. Thus, the term  $U_b Q_b$  in Eq. (2.3) might be as large as  $4000 \text{ ft/sec}^2$  or 125 g's! On the other hand  $Z_b/m$ , the normal acceleration due to the external force (primarily gravity and aerodynamic lift) may have an upper limit of several g's. Hence artificial accelerations which are perhaps 20 to 50 times greater than the actual accelerations are introduced because of the high rotation rates which the body-axes experience. This means much less favorable computer scaling and hence much poorer solution accuracy for a given computer precision. Furthermore, the high-speed dynamics of the rotational equations are coupled into the translational equations, thus placing severe dynamic response requirements on the computer. The use of flight-path axes greatly alleviates these problems. As we shall see in the next section, the flight path axes allow a more efficient calculation of the aerodynamic angle of attack  $\alpha$  and the aerodynamic angle of sideslip  $\beta$  than body axes allow. Using body axes, and assuming that the ambient air mass is not moving relative to the inertial frame used to define  $\vec{V}_p$ , then the following formulas can be used to obtain  $\alpha$ ,  $\beta$ , and velocity magnitude  $V_p$  from the body-axis velocity components  $U_b$ ,  $V_b$ , and  $W_b$ :

$$\tan \alpha = \frac{W_b}{U_b} \quad (2.4)$$

$$\sin \beta = \frac{V_b}{V_p} \quad (2.5)$$

$$V_p = \frac{U_b}{\cos \alpha \cos \beta} = \left( U_b^2 + V_b^2 + W_b^2 \right)^{1/2} \quad (2.6)$$

### 3. Flight-Path Axis Translational Equations

Next consider the flight-path axes  $x_w, y_w, z_w$  shown in Fig. 3.1. These differ from the body axes  $x_b, y_b, z_b$  by the angle of attack  $\alpha$  and the angle of sideslip  $\beta$ , as shown in the figure. To rotate from body axes to flight-path axes one first pitches the body axes about  $y_b$  through  $-\alpha$ . This defines an intermediate axis system  $x_s, y_s, z_s$  called stability axes. One then yaws about  $z_s$ , through  $\beta$ , which defines the flight-path axes  $x_w, y_w$ , and  $z_w$ . Note that  $V_p$  is the  $x_w$  component of vehicle velocity; the  $y_w$  and  $z_w$  components are zero by definition. Let us define the  $x_w, y_w$ , and  $z_w$  components of flight-path angular velocity relative to inertial space by  $P_w, Q_w$ , and  $R_w$ , and the components of external force along  $x_w, y_w$ , and  $z_w$  by  $X_w, Y_w$ , and  $Z_w$ , respectively. Then, since  $U_w \equiv V_p$  and  $V_w = W_w = 0$ , the translational equations (2.1), (2.2) and (2.3) referred to the flight-path axes become

$$m V_p = X_w \quad (3.1)$$

$$m V_p R_w = Y_w \quad (3.2)$$

$$-m V_p Q_w = Z_w \quad (3.3)$$

Solution of these three equations results in total velocity  $V_p$ , flight-path axis yaw rate  $R_w$ , and flight-path axis pitch rate  $Q_w$ .

Next consider the formulas for  $\alpha$  and  $\beta$ . Reference to Fig. 3.1 shows that  $\dot{\alpha}$  is directed along  $y_s$  with a component  $\dot{\alpha} \cos \beta$  along  $y_w$ . Thus  $\dot{\alpha} \cos \beta$  is equal to the difference between body-axis and flight-path axis angular rates along  $y_w$ . Therefore, from Eq. (3.3) we can write

$$\dot{\alpha} \cos \beta = Q_b \cos \beta - P_b^s \sin \beta + \frac{Z_w}{m V_p} \quad (3.4)$$

where  $P_b^s$  is the body-axis (not stability axis) angular rate along  $x_s$  and is given by

$$P_b^s = P_b \cos \alpha + R_b \sin \alpha \quad (3.5)$$

Similarly, reference to Fig. 3.1 shows that  $\dot{\beta}$  is directed along  $z_w$  and is equal to the difference between flight-path axis and body axis angular rates along  $z_w$ . Thus from Eq. (3.2)

$$\dot{\beta} = \frac{Y_w}{m V_p} - R_b^s \quad (3.6)$$

where  $R_b^s$  is the body-axis angular rate along  $z_s$  and is given by

$$R_b^s = -P_b \sin \alpha + R_b \cos \alpha \quad (3.7)$$

(Note that

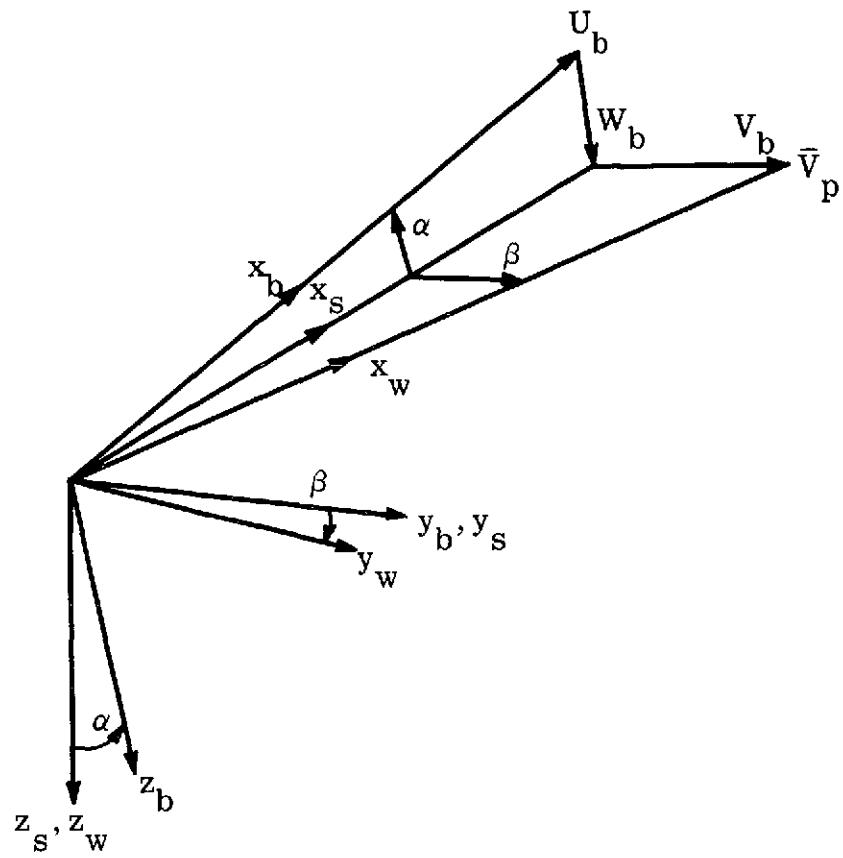


Figure 3.1 Flight Path Axes  $x_w, y_w, z_w$  and the Relation to Body Axes  $x_b, y_b, z_b$ .



$$Q_b^s = Q_b \quad (3.8)$$

since the  $y_b$  body axis and  $y_s$  stability axes are coincident.)

Equations (3.1), (3.4) and (3.5) can be integrated to yield total velocity  $V_p$ , angle of attack  $\alpha$ , and angle of sideslip  $\beta$ . They do not present the scaling difficulty of Eqs. (2.1), (2.2), and (2.3) in the body axes. The body-axis velocity components  $U_b$ ,  $V_b$ , and  $W_b$  can be obtained from  $V_p$ ,  $\alpha$ , and  $\beta$  by the formulas

$$U_b = V_p \cos \alpha \cos \beta \quad (3.9)$$

$$V_b = V_p \sin \beta \quad (3.10)$$

$$W_b = V_p \sin \alpha \cos \beta \quad (3.11)$$

Similarly, the flight-path-axis forces  $X_w$ ,  $Y_w$ , and  $Z_w$  can be derived from body-axis force components  $X_b$ ,  $Y_b$ , and  $Z_b$  by the formulas

$$X_s = X_b \cos \alpha + Z_b \sin \alpha, \quad X_w = X_s \cos \beta + Y_s \sin \beta \quad (3.12)$$

$$Y_s = Y_b, \quad Y_w = -X_s \sin \beta + Y_s \cos \beta \quad (3.13)$$

$$Z_s = -X_b \sin \alpha + Z_b \cos \alpha, \quad Z_w = Z_s \quad (3.14)$$

where  $X_s$ ,  $Y_s$ , and  $Z_s$  are the intermediate stability-axis force components. Frequently the aerodynamic force components are computed along stability axes, in which case the power plant and gravity forces, computed in body axes, would be resolved into stability axes where the

aerodynamic forces are added; then the total forces would be resolved into flight-path axes to allow use of Eqs. (3.1), (3.4), and (3.5). It should be noted that we have assumed throughout this section that the translational and rotational velocity vectors are velocities relative to an inertial reference frame. If the atmosphere through which the vehicle is flying can be considered to be fixed with respect to this inertial frame, then the velocity magnitude  $V_p$  is the vehicle velocity relative to the atmosphere, and  $\alpha$  and  $\beta$  represent the aerodynamic angle of attack and sideslip, respectively. Using the approximation that the earth is flat, and with constant surface winds, it is possible to define the inertial reference frame as a frame attached to the atmosphere. Then all the formulas, as presented, are correct and  $\alpha$ ,  $\beta$  and  $V_p$  can be used for computation of aerodynamic forces and moments. Unfortunately, for a rotating spherical earth axes fixed in the atmosphere are not inertial. If we consider such a frame to be inertial, we will make acceleration errors in Eqs. (3.1), (3.2) and (3.3) of the order of  $\hat{V}^2/r_o$  where  $\hat{V}$  is the vehicle velocity relative to an inertial frame with origin at the center of the earth and  $r_o$  is the radius of the earth. For illustration, consider a vehicle flying eastward in still air with a velocity  $V_a$  relative to the atmosphere. Then  $\hat{V} \cong V_a + r_o \omega_N \cos L$ , where  $\omega_N$  is the earth spin rate and  $L$  is the latitude. If the vehicle is flying westward,  $\hat{V} \cong V_a - r_o \omega_N \cos L$ . For other headings  $\hat{V}$  lies between these values.

For example, if  $V_a = 3000$  ft/sec and  $L = 0$  degrees,  $\hat{V}$  ranges between approximately 1600 and 4400 ft/sec. The corresponding acceleration error in Eqs. (3.1), (3.2) and (3.3), given by  $\hat{V}^2/r_o$ , ranges between approximately 0.1 and 1 ft/sec<sup>2</sup>. For many flight vehicles this is a negligible error. On the other hand for a supersonic transport cruising eastward at 3000 ft/sec this lowers the required steady-state lift by about 3 per cent, which could lower the drag significantly and hence make a noticeable difference in maximum range.

There appears to the authors to be no simple way to take these accelerations into account and still use a flight-path axis system referenced to the atmosphere. One could add an approximate correction acceleration in the vertical direction given by  $V_p^2/r_o$  to the translatory forces in Eqs. (3.1), (3.2), and (3.3). In fact, one could further simplify the computation by adding the term only to Eq. (3.3) based on the argument that most of the time a supersonic aircraft will be in near-level flight at cruise, and that a moderate acceleration error during transient conditions can be accepted.

In this case Eq. (3.3) becomes

$$-m \left( v_p Q_w - \frac{V_p^2}{r_o} \right) = Z_w \quad (3.15)$$

Note that this equation will exhibit acceleration errors of the order of  $V_p^2/r_o$  for conditions far from level flight.

#### 4. Rotational Equations

The only reasonable axes to use for the rotational equations of motion are the body axes. If we let the external moment components along  $x_b$ ,  $y_b$  and  $z_b$  be  $L_b$ ,  $M_b$ , and  $N_b$ , respectively, then summation of moments about the three body axes of a body symmetrical about the  $x_b z_b$  plane leads to the equations:

$$I_{xx} \dot{P}_b - (I_{yy} - I_{zz}) Q_b R_b - I_{xz} (\dot{R}_b + P_b Q_b) = L_b \quad (4.1)$$

$$I_{yy} \dot{Q}_b - (I_{zz} - I_{xx}) R_b P_b + I_{xz} (P_b^2 - R_b^2) = M_b \quad (4.2)$$

$$I_{zz} \dot{R}_b - (I_{xx} - I_{yy}) P_b Q_b - I_{xz} (\dot{P}_b - Q_b R_b) = N_b \quad (4.3)$$

Here  $I_{xx}$ ,  $I_{yy}$ , and  $I_{zz}$  are the moments of inertia about  $x_b$ ,  $y_b$ , and  $z_b$  respectively, and  $I_{xz}$  is the product of inertia of the symmetrical body. Note in Eqs. (4.1), (4.2), and (4.3) that the second term in each equation represents a nonlinear inertial coupling term. For flight vehicles such as large transport aircraft which do not generate relatively high angular rates these terms often can be neglected. For many flight vehicles the roll-rate  $P_b$  has a maximum value which is considerably higher than pitch-rate  $Q_b$  or yaw-rate  $R_b$ . Hence the  $Q_b R_b$  term in Eq. (4.1) often can be neglected in comparison with the  $R_b P_b$  and  $P_b Q_b$  terms in Eqs. (4.2) and (4.3), respectively.

The third term in each of Eqs. (4.1), (4.2), and (4.3) represents the effect of the product of inertia  $I_{xz}$ . If the  $x$ ,  $y$ ,  $z$  body axes have been

chosen to be almost coincident with the principle axes, this term may be negligible in all three equations, since  $I_{xz}$  will be very small compared with the principle moments of inertia. For relatively low angular rates the nonlinear terms ( $P_b Q_b$ ,  $P_b^2 - R_b^2$ , and  $Q_b R_b$ ) usually can be neglected and in any event  $R_b^2$  frequently can be neglected as small compared with  $P_b^2$  in Eq. (4.2).

## 5. Computation of Euler Angles for a Flat Earth

Solution of Eqs. (4.1), (4.2) and (4.3) results in computation of the body-axis angular velocity components  $P_b$ ,  $Q_b$ , and  $R_b$ . These must be integrated a second time to obtain the orientation of the vehicle body axes with respect to the desired reference axes, typically Euler axes which point North, East, and toward the center of the earth ( $x_e$ ,  $y_e$ , and  $z_e$  in Fig. 5.1). This orientation usually is expressed in terms of the conventional aircraft Euler angles, i. e., heading-angle  $\psi$ , pitch angle  $\theta$ , and bank angle  $\phi$ . These angles usually are computed from  $P_b$ ,  $Q_b$  and  $R_b$  by the following well-known equations:

$$\dot{\psi} = (R_b \cos \phi + Q_b \sin \phi) / \cos \theta \quad (5.1)$$

$$\dot{\theta} = Q_b \cos \phi - R_b \sin \phi \quad (5.2)$$

$$\dot{\phi} = P_b + \dot{\psi} \sin \theta \quad (5.3)$$

Note that, since  $P_b$ ,  $Q_b$ ,  $R_b$  are the body axis components of the

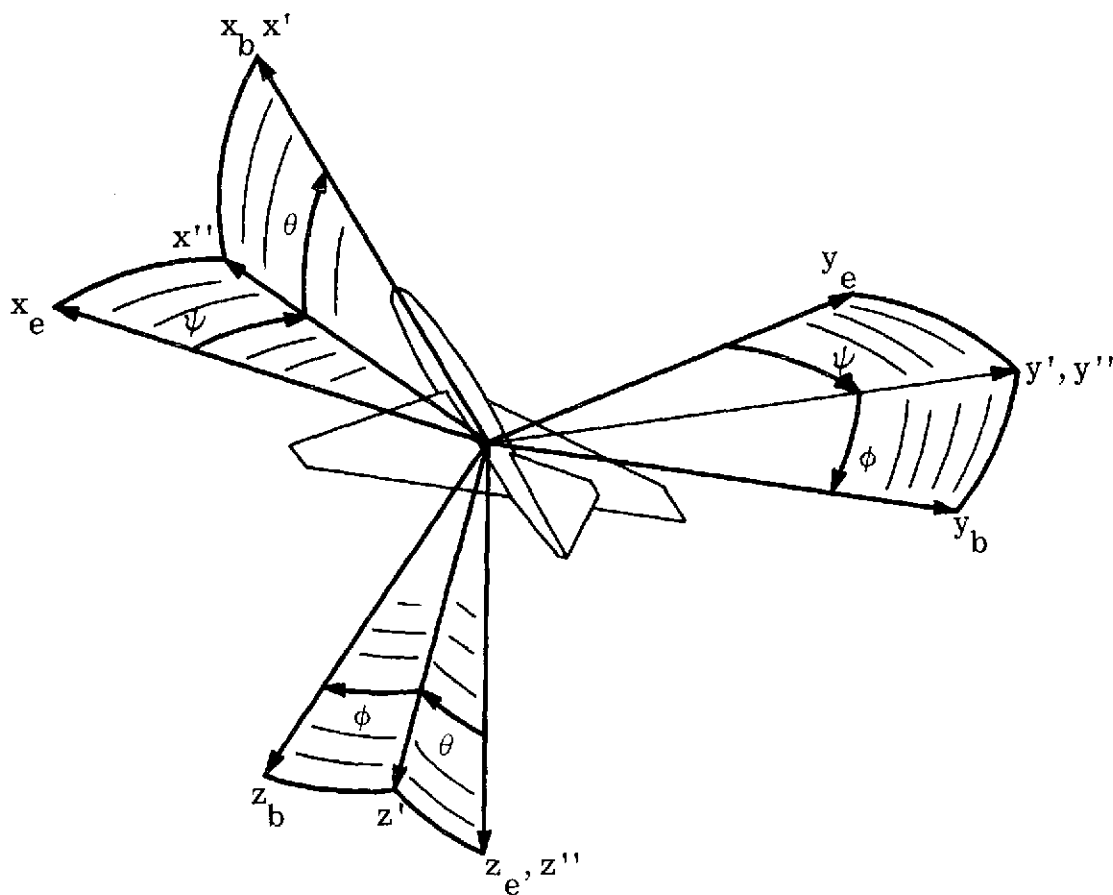


Figure 5.1 Conventional Aircraft Euler Angles.

vehicle angular velocity relative to an inertial reference system, there is a small error introduced by the angular velocity of axes which point North, East, and down relative to a spherical earth. This small error can be corrected, if necessary, using Eqs. (7.1)-(7.3) in the next section.

The well known singularity of the Euler angle system at  $\theta = \pm \pi/2$  can be avoided, if necessary, by computing direction cosines or quaternions [4] instead of Euler angles, or by introducing a fourth angle [3].

The use of quaternions rather than direction cosines should be considered if a system free of singularities is needed, since there are only four quaternions with a single redundancy as compared with nine direction cosines with six redundancies.

## 6. Computation of Vehicle Position for a Flat Earth

Once the orientation of the flight vehicle with respect to the Euler axes has been established, e.g., by means of the Euler angles, then it is possible to compute the velocity north,  $U_e$ ; the velocity east,  $V_e$ , and the velocity downward,  $W_e$ , or its negative, the rate of climb  $\dot{h}$ . Direct integration then yields the vehicle position.

Determination of  $U_e$ ,  $V_e$ ,  $W_e$  is complicated by the fact that the vehicle velocity vector,  $\bar{V}_p$  lies along the x wind axis and therefore must

be resolved from wind to earth axes. Unfortunately, the complete orientation of the wind axes is known only relative to body axes, hence it is necessary to perform the resolution of  $\bar{V}_p$  into earth axes by first resolving it into body axes, then from body axes to earth axes. The resolution of  $\bar{V}_p$  from wind axes to body axes is accomplished by the transformation given in Eqs. (3.9), (3.10), and (3.11).

The resolution of vehicle velocity from body axes to earth axes can be accomplished by using direction cosines. Thus let  $\ell_1$ ,  $\ell_2$ , and  $\ell_3$  be the projections of a unit vector along the x body axis onto the  $x_e$ ,  $y_e$ , and  $z_e$  earth axes, respectively. Similarly, let  $m_1$ ,  $m_2$ , and  $m_3$  be the projections of a unit vector along the y body axis onto the  $x_e$ ,  $y_e$ , and  $z_e$  earth axes, respectively. In the same way let  $n_1$ ,  $n_2$ , and  $n_3$  be the projections of a unit vector the z body axis along the  $x_e$ ,  $y_e$ , and  $z_e$  earth axes, respectively. Then by definition

$$\text{velocity north} = U_e = \ell_1 U_b + m_1 V_b + n_1 W_b \quad (6.1)$$

$$\text{velocity east} = V_e = \ell_2 U_b + m_2 V_b + n_2 W_b \quad (6.2)$$

$$\text{velocity downward} = W_e = -\dot{h} = \ell_3 U_b + m_3 V_b + n_3 W_b \quad (6.3)$$

It is easy to show that the direction cosines are related to Euler angles by the following formulas:



$$\left. \begin{aligned} \ell_1 &= \cos \theta \cos \psi \\ \ell_2 &= \cos \theta \sin \psi \\ \ell_3 &= -\sin \theta \end{aligned} \right\} \quad (6.4)$$

$$\left. \begin{aligned} m_1 &= -\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi \\ m_2 &= \cos \phi \cos \psi + \sin \phi \sin \theta \sin \psi \\ m_3 &= \sin \phi \cos \theta \end{aligned} \right\} \quad (6.5)$$

$$\left. \begin{aligned} n_1 &= \sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi \\ n_2 &= -\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi \\ n_3 &= \cos \theta \cos \phi \end{aligned} \right\} \quad (6.6)$$

(Equivalent formulas for direction cosines in terms of quaternions are given in Ref. 4.) An alternative mechanization of Eqs. (6.1) through (6.6) avoids computation of the direction cosines by instead performing successive resolution of the velocity components  $U_b$ ,  $V_b$ , and  $W_b$  through the angles  $-\phi$ ,  $-\theta$ , and  $-\psi$ . Consider the intermediate axis system  $x'$ ,  $y'$ ,  $z'$  in Fig. 5.1. Clearly the velocity components  $U'$ ,  $V'$ , and  $W'$  along  $x'$ ,  $y'$ , and  $z'$  are given by

$$U' = U_b, \quad V' = V_b \cos \phi - W_b \sin \phi, \quad W' = V_b \sin \phi + W_b \cos \phi \quad (6.7)$$

Defining  $U''$ ,  $V''$ , and  $W''$  as the velocity components along the intermediate axes  $x''$ ,  $y''$ , and  $z''$ , we have

$$U'' = U' \cos \theta + W' \sin \theta, \quad V'' = V', \quad W'' = -U' \sin \theta + W' \cos \theta \quad (6.8)$$

Finally, from Fig. 4.1 we see that

$$U_e = U'' \cos \psi - V'' \sin \psi, \quad V_e = U'' \sin \psi + V'' \cos \psi, \quad W_e = -\dot{h} = W'' \quad (6.9)$$

Successive application of Eqs. (6.7), (6.8), and (6.9) for  $U$ ,  $V$ , and  $W$  to obtain  $U_e$ ,  $V_e$ , and  $W_e$  requires fewer mathematical operations than using Eqs. (8.1) through (8.6). It therefore has computational advantages using either an analog or digital mechanization. In computing ground coordinates from  $U_e$  and  $V_e$  in Eqs. (6.1) and (6.2), or Eq. (6.9), it is important to note that  $U_e$  and  $V_e$  represent airspeed components north and east, respectively. To convert them to groundspeed components the north component of wind,  $w_x$ , must be subtracted from  $U_e$ , and the east component of wind,  $w_y$ , must be subtracted from  $V_e$ , respectively. Thus if  $s_x$  and  $s_y$  represent distance traveled north and east, respectively, then

$$\dot{s}_x = U_e - w_x \quad (6.10)$$

$$\dot{s}_y = V_e - w_y \quad (6.11)$$

Equations (6.10) and (6.11) are valid only for steady winds, since the implicit assumption was made that axes stationary relative to the atmosphere are inertial. Equations (3.1), (3.2), and (3.3) are referred to inertial space and correction terms must be added if the reference axes are not inertial.

## 7. Computation of Vehicle Euler Angles and Position for a Rotating Spherical Earth

In the previous section we presented the formulas for computing vehicle position over a flat earth with steady winds. We can use the same position formulas to obtain velocity north,  $U_e$ , and velocity east,  $V_e$ , over a rotating spherical earth with radius  $r_o$ , angular rate  $\omega_N$ . However,  $U_e$  and  $V_e$  will represent airspeed components and must be corrected to yield groundspeed components  $\dot{s}_x$  and  $\dot{s}_y$  respectively. Furthermore, when a spherical earth is considered it may be necessary to correct the vehicle angular rates used to compute Euler angles in order to take into account the rotating reference frame, as pointed out earlier in Section 6. It can be shown that the body axis components of the vehicle angular velocity relative to the Euler reference frame,  $P_{be}$ ,  $Q_{be}$ ,  $R_{be}$  are given by the following formulas [ 5 ]:

$$P_{be} = P_b - \frac{\dot{s}_y}{r} \ell_1 + \frac{\dot{s}_x}{r} \ell_2 + \frac{\dot{s}_y}{r} \tan L \ell_3 \quad (7.1)$$

$$Q_{be} = Q_b - \frac{\dot{s}_y}{r} m_1 + \frac{\dot{s}_x}{r} m_2 + \frac{\dot{s}_y}{r} \tan L m_3 \quad (7.2)$$

$$R_{be} = R_b - \frac{\dot{s}_y}{r} n_1 + \frac{\dot{s}_x}{r} n_2 + \frac{\dot{s}_y}{r} \tan L n_3 \quad (7.3)$$

Here  $r$  is the radial distance of the vehicle from the center of the earth and is given by

$$r = r_o + h \quad (7.4)$$

where  $r_o$  is the radius of the earth and  $h$  is the vehicle altitude. In many cases we can substitute  $r_o$  for  $r$  in Eqs. (7.1), (7.2), and (7.3) and still obtain sufficient accuracy.

The values of  $P_{be}$ ,  $Q_{be}$ , and  $R_{be}$  in these equations are then used to compute the Euler angle rates. Thus by analogy with Eqs. (5.1), (5.2), and (5.4)

$$\dot{\psi} = (R_{be} \cos \phi + Q_{be} \sin \phi) / \cos \theta \quad (7.5)$$

$$\dot{\theta} = Q_{be} \cos \phi - R_{be} \sin \phi \quad (7.6)$$

$$\dot{\phi} = P_{be} + \dot{\psi} \sin \theta \quad (7.7)$$

In many cases the computations involved in Eqs. (7.1), (7.2), and (7.3) can be neglected, i. e., we can assume that  $P_{be} \cong P_b$ ,  $Q_{be} \cong Q_b$ ,  $R_{be} \cong R_b$ . This is particularly true if the overall six-degree-of-freedom computation involves a control system (automatic or human) which attempts to maintain  $\psi$ ,  $\theta$ , and  $\phi$  at specified values. In any case the correction rates are the order of  $V_p / r_o$ . For example, if  $V_p = 2000$  ft/sec, the correction rate is equal to approximately 0.005 degrees per second. On the other hand, if the flight-vehicle problem includes a stable platform, the rate corrections given by Eqs. (7.1), (7.2), and (7.3) may be important.

It should be noted that  $\dot{s}_x$  and  $\dot{s}_y$  in Eqs. (7.1), (7.2), and (7.3) represent vehicle velocity components north and east, respectively, over the surface of a non-rotating earth with steady winds. On the other hand Eqs. (6.1) and (6.2), or alternatively, Eqs. (6.9), gives us  $U_e$  and  $V_e$ , i. e., vehicle velocity components north and east, respectively, relative to the inertial reference frame for the translational equations of motion. We made the approximation in Section 3 that this reference frame is fixed relative to the ambient atmosphere. We can account for the linear velocity of the atmospheric reference frame (but not the angular velocity) by noting that relative to the surface of a non-rotating earth it is moving northward with the northerly component of wind and eastward with the sum of the rate due to earth spin and that due to eastward component of wind. Thus we can write the following equations:

$$\dot{s}_x = U_e - w_x \quad (7.8)$$

$$\dot{s}_y = V_e + \omega_N r \cos L - w_y \quad (7.9)$$

It can also be shown [ 5] that the time rate of change of latitude and longitude are given by the following formulas:

$$\dot{L} = \frac{\dot{s}_x}{r} \quad (7.10)$$

$$\dot{\lambda} = \frac{\dot{s}_y}{r \cos L} - \omega_N \quad (7.11)$$

It should be noted that motion of a flight-vehicle over a rotating earth can be treated exactly [ 5 ], but that the exact translational equations referred to axes fixed relative to the ambient atmosphere, are very complicated. Thus many of the computer mechanization advantages for flight-path axes are lost, and the computation of  $\alpha$ ,  $\beta$ , and  $V_p$  is much less elegant. We have attempted in this section to describe how one can utilize the flight-path axis system and still correct approximately for the fact that the vehicle is flying over a rotating earth with surface winds. This approach should be adequate for all but the most exacting requirements unless the vehicle reaches hypersonic speeds. For subsonic vehicles or supersonic vehicles traveling over relatively short distances the flat-earth equations in Sections 3 through 6 should be adequate.

#### 8. Computation of Aerodynamic Forces and Moments

Aerodynamic forces and moments for flight vehicles normally are computed in stability axes. Thus let  $X_a$ ,  $Y_a$ , and  $Z_a$  be the aerodynamic forces along the  $x_s$ ,  $y_s$ , and  $z_s$  stability axes shown in Fig. 3.1. Here  $-X_a$  corresponds to drag  $D$  and  $-Z_a$  corresponds to lift  $L$ .

In addition to aerodynamic forces the flight vehicle experiences propulsion and gravity forces. These are most conveniently defined in body axes. Let us denote the propulsion force components along the  $x$  and  $z$  body axes by  $X_p$  and  $Z_p$ , respectively. The gravity force

components along x, y, and z will by definition be  $mg\ell_3$ ,  $mgm_3$ , and  $mg n_3$  respectively. The sum of the propulsive forces and gravity forces along body axes can then be resolved to stability axes, where the aerodynamic forces are added to obtain the total force component  $X_s$ ,  $Y_s$ , and  $Z_s$  along the stability axes. Using (6.4), (6.5), and (6.6) to express the direction cosines  $\ell_3$ ,  $m_3$ ,  $n_3$  in terms of Euler angles, we obtain:

$$X_s = (X_p - mg \sin \theta) \cos \alpha + (Z_p + mg \cos \theta \cos \phi) \sin \alpha - D \quad (8.1)$$

$$Y_s = mg \cos \theta \sin \phi + Y_a \quad (8.2)$$

$$Z_s = - (X_p - mg \sin \theta) \sin \alpha + (Z_p + mg \cos \theta \cos \phi) \cos \alpha - L \quad (8.3)$$

Finally, reference to Fig. 3.1 shows that the force components  $X_w$ ,  $Y_w$ , and  $Z_w$  along the flight-path axes can be computed using the following formulas:

$$X_w = X_s \cos \beta + Y_s \sin \beta \quad (8.4)$$

$$Y_w = - X_s \sin \beta + Y_s \cos \beta \quad (8.5)$$

$$Z_w = Z_s \quad (8.6)$$

These force components are used to mechanize the translational equations given in Section 3. If the aerodynamic forces are given in body axes rather than stability axes, the modification of Eqs. (8.1), (8.2), and (8.3) is apparent.

The external moments acting on the flight vehicle consist of aerodynamic moments  $L_a$ ,  $M_a$ , and  $N_a$ , normally given in stability axes, and power plant moments  $L_p$ ,  $M_p$ , and  $N_p$ , given in body axes. The total moments  $L$ ,  $M$ , and  $N$  in body axes become the following:

$$L = L_a \cos \alpha - N_a \sin \alpha + L_p \quad (8.7)$$

$$M = M_a + M_p \quad (8.8)$$

$$N = L_a \sin \alpha + N_a \cos \alpha + N_p \quad (8.9)$$

Again, if the aerodynamic moments are given in body axes, the simplification of Eqs. (8.7), (8.8), and (8.9) is obvious (set  $\alpha = 0$ ).

Figure 8.1 shows a block diagram of the over-all six-degree-of-freedom equations for the case of a flat earth.



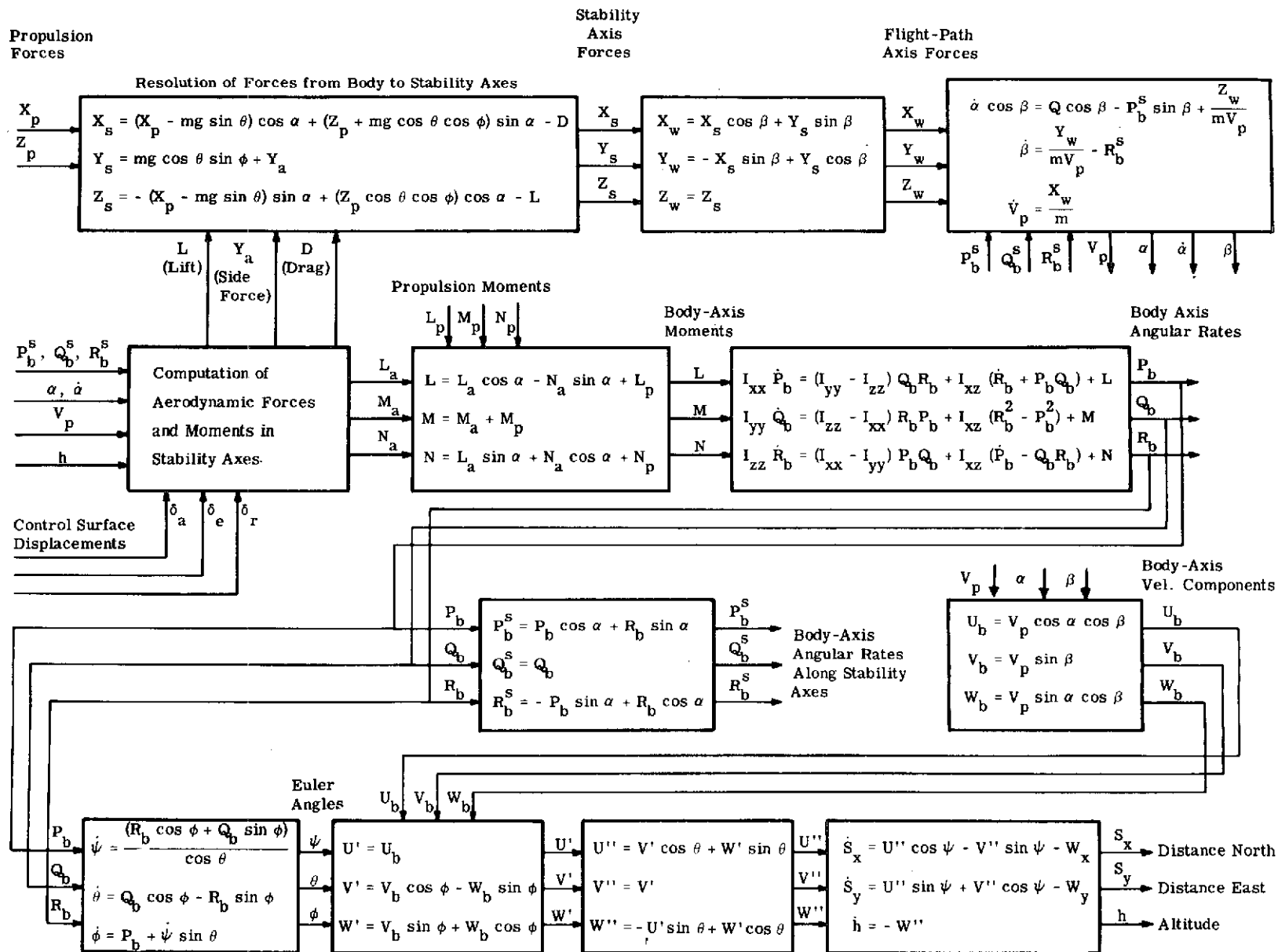


Figure 8.1 Block Diagram of Combined Flight-Path Axis, Body-Axis System for a Flat Earth, Steady Winds.

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